

Stationary black holes and attractor mechanism

Dumitru Astefanesei¹ and Hossein Yavartanoo²

¹*Global Edge Institute, Tokyo Institute of Technology
Tokyo 152-8550, JAPAN*

²*Center for Theoretical Physics and BK-21 Frontier Physics Division Seoul National
University, Seoul 151-747 KOREA*

ABSTRACT

We investigate the symmetries of the near horizon geometry of extremal stationary black hole in four dimensional Einstein gravity coupled to abelian gauge fields and neutral scalars. Careful consideration of the equations of motion and the boundary conditions at the horizon imply that the near horizon geometry has $SO(2,1) \times U(1)$ isometry. This complements the rotating attractors proposal of hep-th/0606244 that had assumed the presence of this isometry. The extremal solutions are classified into two families differentiated by the presence or absence of an ergo-region. We also comment on the attractor mechanism of both branches.

¹E-mail: dumitru@th.phys.titech.ac.jp

²E-mail: yavar@phya.snu.ac.kr;

1 Introduction

The attractor mechanism plays a key role in understanding the entropy of non-supersymmetric extremal black holes in string theory [1, 2]. In certain cases, the macroscopic entropy of extremal non-supersymmetric attractor horizons can be matched to the weak coupling statistical entropy despite the fact that these quantities do not seem to be protected by supersymmetry [3, 4, 5, 6, 7].

It was originally noticed in [8] that the extremal four dimensional Kerr and Kerr-Newman black holes have an $SO(2,1) \times U(1)$ isometry. The last year, the authors of [9], found even more four dimensional extremal black holes had this isometry. Emboldened by this observation, they found that, for four dimensional stationary extremal black holes, in a theory of gravity with neutral scalar fields non-minimally coupled to abelian gauge fields, one can generalise the entropy function formalism of [10] simply by assuming an $SO(2,1) \times U(1)$ near horizon geometry.

The generalised entropy function is constructed, on an $SO(2,1) \times U(1)$ symmetric background, by taking the Legendre transform (with respect to the electric charges and angular momentum) of the reduced Lagrangian evaluated at the horizon. Extremising the entropy function is equivalent to the equations of motion and its extremal value corresponds to the entropy. Since the entropy function depends only on the near horizon geometry, its extremum and hence the entropy is independent of the asymptotic data. This is precisely the attractor behaviour. However, if the entropy function has flat directions something interesting happens: while the extremum remains fixed, flat directions will not be fixed by near horizon data and can depend on the asymptotic moduli.

There exist two distinct branches of stationary extremal black hole solutions which, in [9], are dubbed ‘ergo-’ and ‘ergo-free’ branches according to their properties.¹ The first branch, also known as the fast branch, can exist for angular momentum of magnitude larger than a certain lower bound and does have an ergo-region. On the other hand, the ergo-free branch can exist only for angular momentum of magnitude less than a certain upper bound. The ergo-free branch can also be smoothly connected to a static extremal black hole.

The entropy function has no flat directions for the ergo-free branch: the scalar and all other background fields at the horizon are independent of the asymptotic data. However, there is a drastic change for the ergo-branch — the entropy function has flat directions: despite the entropy being independent of the moduli, the near horizon fields are dependent on the asymptotic data.

We find it significant that, the existence of an ergo-region allows energy to be extracted classically either by the Penrose process for point particles or by superradiant scattering for fields. It is tempting to believe that the presence of the ergo-sphere is intimately related to the appearance of flat directions. One might say that the ergo-branch, not completely isolated

¹The existence of two branches in the moduli space of extremal rotating black holes was discussed for the first time in [11].

from its environment due to these processes, retains some dependence on the asymptotic moduli. From this perspective, it is amazing that the black hole is isolated enough for the entropy to remain independent.²

A consistent microscopic picture for Kaluza-Klein (KK) black hole in agreement with the macroscopic analysis of rotating attractors [9] was provided in [3, 4]. That is, the D-brane model reproduces the entropy of KK black hole, while the *mass gets renormalized* from weak to strong coupling *just* for the *ergo-branch* black hole solutions in agreement with the existence of the flat directions in the entropy function for this branch. Emparan and Maccarrone, [3], have also provided a microscopic interpretation for the superradiant ergosphere — even if the temperature is vanishing, the extremal black holes with ergosphere correspond to states with both left- and right-moving excitations such that the open strings can combine and the emission of closed strings is possible. The extraction of energy should reduce the angular momentum in such a way that the event horizon area is increasing (it can not decrease in the classical processes). Indeed, since the left-moving excitations have spin, the emitted closed string will necessarily carry angular momentum away from the black hole.

In this note we fill up a gap in the proposal of [9] by proving that the near horizon geometry of extremal rotating black holes in Einstein-Hilbert gravity coupled to abelian gauge fields and neutral scalar fields has an enhanced $SO(2, 1) \times U(1)$ symmetry. Unlike the static case where the near horizon geometry is $AdS_2 \times S^2$, the AdS_2 part does not decouple from the angular part and the values of the moduli at the horizon have an angular dependence. Also, by adding angular momentum to static black holes, the $SO(3)$ symmetry of the sphere is broken to $U(1)$. However, the near horizon geometry is still universal in the sense that is still independent of the coupling constants and is determined just by charges and angular momentum parameter. The attractor mechanism is related to the extremality rather than to the supersymmetry property of the theory/solution. Indeed, the enhanced symmetry of the near horizon geometry and the long throat of AdS_2 is at the basis of the attractor mechanism for stationary black holes [9, 10].

2 Generalities

We consider a theory of gravity coupled to a set of massless scalars and vector fields, whose general bosonic action has the form

$$I[G_{\mu\nu}, \phi^i, A_\mu^I] = \frac{1}{k^2} \int_M d^4x \sqrt{-G} [R - 2g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j - f_{AB}(\phi) F_{\mu\nu}^A F^{B\mu\nu} - \frac{1}{2\sqrt{-G}} \tilde{f}_{AB}(\phi) F_{\mu\nu}^A F_{\rho\sigma}^B \epsilon^{\mu\nu\rho\sigma}], \quad (1)$$

where $F_{\mu\nu}^A$ with $A = (0, \dots, N)$ are the gauge fields, ϕ^i with $(i = 1, \dots, n)$ are the scalar fields, and $k^2 = 16\pi G_4$. The moduli determine the gauge coupling constants and $g_{ij}(\phi)$ is the metric

²It is possible that the addition of higher derivative terms might lift these flat directions. It would be interesting to see whether this would erase the ergo-sphere.

in the moduli space. We use Gaussian units to avoid extraneous factors of 4π in the gauge fields, and the Newton's constant is set to $G_4 = 1$.

Varying the action we obtain the following equations of motion for the metric, moduli, and the gauge fields:

$$R_{\mu\nu} - 2g_{ij}\partial_\mu\phi^i\partial_\nu\phi^j = f_{AB}\left(2F_{\mu\lambda}^A F_{\nu}^{B\lambda} - \frac{1}{2}G_{\mu\nu}F_{\alpha\lambda}^A F^{B\alpha\lambda}\right) \quad (2)$$

$$\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}g_{ij}\partial^\mu\phi^j) = \frac{1}{4}\frac{\partial f_{AB}}{\partial\phi^i}F_{\mu\nu}^A F^{B\mu\nu} + \frac{1}{8\sqrt{-G}}\frac{\partial\tilde{f}_{AB}}{\partial\phi^i}F_{\mu\nu}^A F_{\rho\sigma}^B \epsilon^{\mu\nu\rho\sigma} \quad (3)$$

$$\partial_\mu\left[\sqrt{-G}\left(f_{AB}F^{B\mu\nu} + \frac{1}{2\sqrt{-G}}\tilde{f}_{AB}F_{\rho\sigma}^B \epsilon^{\mu\nu\rho\sigma}\right)\right] = 0. \quad (4)$$

To get the equations of motion, we have varied the moduli and the gauge fields independently. The Bianchi identities for the gauge fields are $F^A_{[\mu\nu;\lambda]} = 0$.

We are interested in stationary black hole solutions to the equations of motion. In general relativity the boundary conditions are fixed. However, in string theory one can obtain interesting situations by varying the asymptotic values of the moduli and so, in general, the asymptotic moduli data should play an important role in characterizing these solutions. Indeed, the non-extremal black hole solutions are characterized by the usual conserved charges and also by the scalar charges — the scalar charge is defined as the monopole in the multipoles expansion of the scalar field at the boundary. Thus all its properties are moduli dependent, e.g. the entropy depends by the asymptotic values of the moduli. However, the entropy of extremal solutions obtained by taking the smooth limit when the temperature is vanishing is independent of the asymptotic moduli data. We will see in the next section that the enhanced symmetry of their near horizon geometry make them special in this regard.

Now let us write down the most general stationary black hole solution by using just its symmetries.³ An asymptotically flat spacetime is stationary if and only if there exists a Killing vector field, ξ , that is time-like at spatial infinity — it can be normalized such that $\xi^2 = -1$. It was also been shown that *stationarity* implies *axisymmetry* [12] and so the event horizon is a Killing horizon. Using the time-independence and axisymmetry we can write the most general stationary metric with an ‘axial’ Killing vector, ∂_ϕ , as

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt d\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2. \quad (5)$$

The event horizon of a stationary black hole is a Killing horizon of $\partial_t + \omega\partial_\phi$, where the constant coefficient ω is the angular velocity of the horizon. It is convenient to rewrite the

³The thermodynamics of the non-extremal black hole solutions using the method developed in [13, 14, 15] will be presented in [16].

metric (5) in the ADM form

$$ds^2 = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt) = -N^2 dt^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \quad (6)$$

and so we obtain:

$$N^2 = \frac{(g_{t\phi})^2}{g_{\phi\phi}} - g_{tt}, \quad N^\phi = \frac{g_{t\phi}}{g_{\phi\phi}}, \quad \gamma_{ij} = g_{ij}.$$

The shift vector, N^ϕ , evaluated at the horizon reproduces the angular velocity of the horizon:

$$\omega = -N^\phi \Big|_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \Big|_H.$$

By eliminating the conical singularity in the Euclidean $(\tau = it, r)$ sector, we obtain the temperature

$$T = \frac{1}{\Delta\tau} = \frac{(N^2)'}{4\pi\sqrt{N^2 g_{rr}}} \Big|_H. \quad (7)$$

3 Near horizon geometry of extremal black holes

We consider a generic covariant two derivative gravity Lagrangian that has three basic components: metric, scalars, and gauge fields. We show that, given a few simple assumptions, the near horizon geometry of a stationary, extremal spinning black hole solutions of this Lagrangian necessarily has the near horizon symmetry $SO(2, 1) \times U(1)$. To prove the previous statement, we make use of the following ingredients:

- Symmetries: we assume time independence and axisymmetry;
- The black hole is extremal — in other words the surface gravity (temperature) is zero;
- We expand the fields near the horizon and take a scaling limit;
- Gauge choices;
- Finiteness of certain physical quantities;
- Equations of motion;
- Spherical topology of the horizon.

3.1 Constraining the metric

As a warm-up exercise, we begin by examining 4-dimensional spherically symmetric black holes by using the following ansatz:

$$ds^2 = -a(r)^2 dt^2 + a(r)^{-2} dr^2 + b(r)^2 d\Omega^2. \quad (8)$$

The near horizon geometry of the extremal black holes can be obtained in two steps: first, take the extremal limit when the temperature is vanishing (this is a smooth limit on the Lorentzian

section) and then obtain the near horizon geometry. We expand the metric components near the outer horizon and for the non-extremal solution ($r_+ \neq r_-$) we obtain:

$$a^2 = \rho f(r) = \rho(f_0 + f_1\rho + f_2\rho^2 + \dots), \quad b^2 = \frac{\rho(\rho + \epsilon)}{a^2} = \frac{\rho + \epsilon}{f_0 + f_1\rho + f_2\rho^2 + \dots}. \quad (9)$$

and so the temperature is $f_0 = 4\pi T$. Here we used a coordinates system such that the horizon is at $\rho = r - r_+ = 0$ and defined the non-extremality parameter $\epsilon = r_+ - r_-$.

The extremal limit is obtained for $4\pi T = f_0 \rightarrow 0$ and to obtain the near horizon geometry we also take $\rho = 0$. By changing the coordinate $\tau = t/f_1$ one can easily obtain the $AdS_2 \times S^2$ explicitly

$$ds^2 = \frac{1}{f_1}(-\rho^2 d\tau^2 + \frac{1}{\rho^2} d\rho^2) + \frac{1}{f_1} d\Omega^2. \quad (10)$$

We use a similar method to obtain the near horizon geometry of stationary extremal black holes. However, the extremal limit in this case is more subtle since we should also consider the non-diagonal component ($\sim d\phi dt$) of the metric. Let us first rewrite the metric components in a more useful form:

$$N^2 = (r - r_+)(r - r_-)\mu(r, \theta), \quad N^\phi = -\omega + (r - r_+)\eta(r, \theta), \quad g_{rr} = \frac{1}{(r - r_+)(r - r_-)\Lambda(r, \theta)}, \quad (11)$$

where $\mu(r, \theta)$, $\eta(r, \theta)$, and $\Lambda(r, \theta)$ are regular functions. The temperature can be read off as before and using eq. (7) we obtain

$$T = \frac{r_+ - r_-}{4\pi} \sqrt{\mu(\theta)\Lambda(\theta)}, \quad (12)$$

where $\mu(\theta)$ and $\Lambda(\theta)$ are the values of $\mu(r, \theta)$ and $\Lambda(r, \theta)$ at the outer horizon. For a non-extremal black hole the temperature (surface gravity) is finite and constant on the horizon and so we obtain $\sqrt{\mu(\theta)\Lambda(\theta)} = C$, where C is a constant that depends on the charges P, Q and the (mass and angular) parameters m, a . Expanding the ADM form of the metric near the outer horizon we obtain the following metric:

$$-(r - r_+)(r - r_-)\mu(\theta)dt^2 + g_{\phi\phi}[d\phi + (r - r_+)\eta(\theta)dt]^2 + \frac{1}{(r - r_+)(r - r_-)\Lambda(\theta)}dr^2 + g_{\theta\theta}d\theta^2. \quad (13)$$

To obtain the near horizon geometry, we first construct the following family of metrics

$$r \rightarrow r_+ + \lambda r, \quad t \rightarrow \frac{t}{\lambda}, \quad (14)$$

where λ is an arbitrary parameter. There is a smooth limit $\lambda \rightarrow 0$ for which the near horizon geometry is obtained. Obviously, this is important for stationary field configurations where there exist also terms of the form $dr dt$. This limit is especially useful when we consider the near horizon expansion of the gauge fields.

Taking the extremal limit ($r_+ \rightarrow r_-$) and choosing a particular gauge we obtain

$$ds^2 = \mu(\theta) \left(-r^2 dt^2 + \frac{dr^2}{C^2 r^2} \right) + \frac{\sin^2 \theta}{\mu(\theta)} (d\tilde{\phi} + r\eta(\theta)dt)^2 + \frac{\mu(\theta)}{C^2} d\theta^2, \quad (15)$$

where $\tilde{\phi} = \phi - \frac{\omega t}{\lambda}$. To obtain the above expression we use the appropriate coordinates system in which $g_{\theta\theta} = r^2 g_{rr}$ and the gauge freedom to write $\mu(\theta) g_{\phi\phi}(\theta) = \sin^2 \theta$. Here, we have considered the metric in a rotating frame with respect to a distant observer with the angular velocity equal to that of the black hole. For a horizon with spherical topology, we require

$$\frac{\sin^2 \theta}{C^2 \mu^2(\theta)} \sim \begin{cases} \theta^2 & \theta \rightarrow 0 \\ (\pi - \theta)^2 & \theta \rightarrow \pi \end{cases} \quad (16)$$

such that the deformed horizon, labelled by the coordinates (θ, ϕ) , is a smooth deformation of the sphere. Unlike the static case, the fields at the horizon have an angular dependence and so solving the attractor equations requires boundary conditions, i.e. the values of the fields at the poles of the horizon.

Let us end up this subsection with an important comment about the extremal limit. For a stationary black hole there are three intensive parameters associated to the horizon: the angular velocity, the temperature, and the electric (magnetic) potential. Thus, there are two interesting extremal limits $T = 0$ when the angular velocity is or is not vanishing. In the discussion section we comment further on the physics of the extremal black holes.

3.2 Constraining the scalars and gauge fields

Let us start by investigating the scalar and the gauge fields configuration in the near horizon limit. For simplicity, we do not carry on the moduli and gauge fields indices in this subsection — we specialize to one scalar and one gauge field configuration, but the generalization to a configuration with more than one scalar and one gauge field is straightforward. Expanding the scalars at the horizon, $r = 0$, we obtain

$$\phi(r, t) = r^\alpha (\phi(\theta) + r\phi_1(\theta) + \mathcal{O}(r^2) + \dots). \quad (17)$$

Requiring that the scalars are finite at the horizon, implies $\alpha \geq 0$ and then by taking the scaling limit, $r \rightarrow \lambda r$, $\lambda \rightarrow 0$, we find

$$\phi = \begin{cases} \phi_0(\theta) & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \quad (18)$$

in the near horizon region. We assume that the near horizon effective gauge coupling $f(\phi(\theta))$ is well behaved and can be Taylor expanded around the poles, i.e.

$$f(\phi(\theta)) = f_0 + f_1 \theta + \mathcal{O}(\theta^2) \quad (19)$$

Let us turn to the gauge fields and perform a similar analysis. We impose that the gauge fields are time-independent and start with the following ansatz

$$A = A_t(r, \theta)dt + A_r(r, \theta)dr + A_\theta(r, \theta)d\theta + A_\phi(r, \theta)d\phi, \quad (20)$$

that can be further simplified by choosing an appropriate gauge choice to fix $A_\theta = 0$ (or $A_r = 0$). We can expand the gauge fields about the horizon as

$$A = r^\alpha [a_t(\theta) + \mathcal{O}(r)] rdt + r^\beta [a_r(\theta) + \mathcal{O}(r)] \frac{dr}{r} + r^\gamma [a_\phi(\theta) + \mathcal{O}(r)] d\phi \quad (21)$$

Requiring F^2 remains finite at the horizon implies $\alpha, \beta, \gamma \geq 0$. We take $\alpha, \beta, \gamma = 0$ so that after taking the scaling limit $r \rightarrow \lambda r$, $t \rightarrow t/\lambda$, $\lambda \rightarrow 0$ we obtain a non-zero result. With this assumption the scaling limit gives

$$A = a_t(\theta)rdt + a_r(\theta)\frac{dr}{r} + a_\phi(\theta)d\phi + \mathcal{O}(\lambda) \quad (22)$$

The Einstein equations can be written as

$$R_{\mu\nu} - 2\partial_\mu\phi\partial_\nu\phi = f \left(2F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{2}g_{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda} \right) \quad (23)$$

The $(r\theta)$ equation plays an important role in what follows: using the results from the appendix and the fact that $\zeta = 0$, we get

$$\frac{\sin^2(\theta)}{\mu^2(\theta)}\eta(\theta)\eta'(\theta) = 0 \quad (24)$$

which implies $\eta(\theta)$ is, in fact, a constant. This was the last step in our proof — it is straightforward to check that the metric (15) with $\eta(\theta)$ a constant function has the $SO(2, 1) \times U(1)$ isometry.

4 Attractor mechanism

In this section, we consider the attractor mechanism for static and stationary black holes. For the static black hole solutions, we show the equivalence of the entropy function formalism and the effective potential method. Entropy function formalism was generalized to stationary black holes in [9]. We comment on the role played by the enhanced symmetry of the near horizon geometry in decoupling the moduli equations of motion at the horizon from the bulk.

4.1 Static black holes

The Bianchi identity and equation of motion for the gauge fields can be solved by a field strength of the form

$$F^A = f^{AB}(Q_B - \tilde{f}_{BC}P^C)\frac{1}{b^2}dt \wedge dr + P^A \sin\theta d\theta \wedge d\phi, \quad (25)$$

where P^A, Q_A are constants that determine the magnetic and electric charges carried by the gauge field F^A , and f^{AB} is the inverse of f_{AB} .

As discussed in [17], the equations of motion for the moduli are

$$\partial_r(a^2 b^2 g_{ij} \partial_r \phi^j) = \frac{1}{2b^2} \frac{\partial V_{eff}}{\partial \phi^i}, \quad (26)$$

where $V_{eff}(\phi^i)$ is a function of scalar fields ϕ^i given by

$$V_{eff}(\phi_i) = f^{AB}(Q_A - \tilde{f}_{AC} P^C)(Q_B - \tilde{f}_{BD} P^D) + f_{AB} P^A P^B. \quad (27)$$

It is clear from the equation (26) that $V_{eff}(\phi^i)$ is an ‘effective potential’ for the scalar fields — it plays an important role in describing the attractor mechanism [17, 18].

For the attractor mechanism it is sufficient for two conditions to be met. First, for fixed charges, as a function of the moduli, V_{eff} must have a critical point. Denoting the critical values for the scalars as $\phi^i = \phi_0^i$ we have,

$$\partial_i V_{eff}(\phi_{i0}) = 0. \quad (28)$$

Second, the matrix of second derivatives of the potential at the critical point,

$$M_{ij} = \frac{1}{2} \partial_i \partial_j V_{eff}(\phi_{i0}) \quad (29)$$

should have positive eigenvalues.

Once the two conditions mentioned above are met it was argued in [17] that the attractor mechanism works and the entropy is given by the effective potential at the horizon.

The near horizon geometry is $AdS_2 \times S^2$ and so we can apply Sen’s entropy function [10] to investigate the attractor behaviour of static extremal solutions. All other background fields respect the $SO(2,1) \times SO(3)$ symmetry of $AdS_2 \times S^2$. We keep the analysis general in order to understand the role of V_{eff} .

In [10], Sen found that the entropy of a spherically symmetric extremal black hole is the Legendre transform of the Lagrangian density — the only requirements are gauge and general coordinate invariance of the action. In fact, this is similar with a generalization of the Wald’s formalism for extremal black holes and it is based on the observation that there is a *smooth* extremal limit on the Lorentzian section of a charged black hole.

The entropy function is defined as

$$F(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = 2\pi(e_i q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})) = 2\pi(e_i q_i - \int d\theta d\phi \sqrt{-GL}), \quad (30)$$

where $d\phi \sqrt{-GL}$ is the Lagrangian density, $q_i = \partial f / \partial e_i$ are the electric charges, u_s are the moduli values at the horizon, p_i and e_i are the near horizon radial magnetic and electric fields,

and v_1, v_2 are the sizes of AdS_2 and S^2 , respectively. Thus, $F/2\pi$ is the Legendre transform of f with respect to the variables e_i . Then, for an extremal black hole of electric charge \vec{Q} and magnetic charge \vec{P} , Sen have shown that the equations determining \vec{u}, \vec{v} and \vec{e} are given by:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_i} = 0, \quad \frac{\partial F}{\partial e_i} = 0. \quad (31)$$

Thus, the black hole entropy is given by $S = F(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ at the extremum (31). We observe that the entropy function, $F(\vec{u}, \vec{v}, \vec{e}, \vec{p})$, determines the sizes v_1, v_2 of AdS_2 and S_2 , and also the near horizon values of moduli u_s and gauge field strengths e_i .

Now, we are ready to apply this method to our action (1). The general metric of $AdS_2 \times S^2$ can be written as

$$ds^2 = v_1(-\rho^2 d\tau^2 + \frac{1}{\rho^2} d\rho^2) + v_2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (32)$$

The field strength ansatz is

$$F^A = F_{r\tau}^A dr \wedge d\tau + P^A \sin \theta d\theta \wedge d\phi = e^A dr \wedge d\tau + P^A \sin \theta d\theta \wedge d\phi \quad (33)$$

and so

$$\begin{aligned} F(v_1, v_2, e, q, p) &= 2\pi[q_A e^A - f(v_1, v_2, e, p)] \\ f(v_1, v_2, e, p) &= \frac{8\pi}{k^2} \left[-v_2 + v_1 - f_{AB} \left(\frac{-v_2}{v_1} e^A e^B + \frac{v_1}{v_2} p^A p^B \right) - 2\tilde{f}_{AB} e^A e^B \right] \end{aligned} \quad (34)$$

The attractor equations are:

$$\frac{\partial F}{\partial v_1} = 0 \Rightarrow 1 - \frac{v_2}{v_1^2} f_{AB} e^A e^B - \frac{1}{v_2} f_{AB} p^A p^B = 0 \quad (35)$$

$$\frac{\partial F}{\partial v_2} = 0 \Rightarrow -1 + \frac{1}{v_1} f_{AB} e^A e^B - \frac{v_1}{v_2^2} f_{AB} p^A p^B = 0 \quad (36)$$

$$\frac{\partial F}{\partial \phi^i} = 0 \Rightarrow \frac{\partial f_{AB}}{\partial \phi^i} (p^A p^B - e^A e^B) = 2 \frac{\tilde{f}_{AB}}{\partial \phi^i} e^A p^B \quad (37)$$

$$\frac{\partial F}{\partial e^A} = 0 \Rightarrow q_A = \frac{16\pi}{k^2} \left(\frac{v_2}{v_1} f_{AB} e^B - \tilde{f}_{AB} p^B \right) \quad (38)$$

By combining the first two equations we obtain $v = v_1 = v_2 = f_{AB}(e^A e^B + p^A p^B)$ that is expecting also from our near horizon geometry analysis above. It's also easy to check that the entropy is given by F at the attractor critical point:

$$S = F = \frac{16\pi^2}{k^2} f_{AB}(e^A e^B + p^A p^B) = \pi v \quad (39)$$

Using the electromagnetic field ansatz,(25), it can be easily shown that $S = \pi V_{eff}$, $q_A = -Q_A$ (the $-$ sign appears because of our convention for F_{tr}^A), and (38) matches the critical point condition of V_{eff} .

4.2 Stationary black holes

We have shown in the previous section that the near horizon geometry of extremal spinning black holes has the symmetries of $AdS_2 \times S^1$ and can be written as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1(\theta) \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 d\theta^2 + \beta^2 v_2(\theta) (d\phi - \alpha r dt)^2 \quad (40)$$

and the most general field configuration consistent with the $SO(2,1) \times U(1)$ symmetry of $AdS_2 \times S^1$ is of the form:

$$\begin{aligned} \Phi^i &= u^i(\theta) \\ \frac{1}{2} F_{\mu\nu}^A dx^\mu \wedge dx^\nu &= (e^A - \alpha b^A(\theta)) dr \wedge dt + \partial_\theta b^A(\theta) d\theta \wedge (d\phi - \alpha r dt), \end{aligned} \quad (41)$$

where α , β and e_i are constants, and v_1 , v_2 , u^i , and b^A are functions of θ . Here ϕ is a periodic coordinate with period 2π and θ takes value in the range $0 \leq \theta \leq \pi$.

Based on this observation, a generalized entropy function was proposed in [9]

$$F \equiv 2\pi(J\alpha + Q_A e^A - \int d\theta d\phi \sqrt{-\det GL}), \quad (42)$$

and so there is one more attractor equation associated to the angular momentum J . Thus, the entropy and the near horizon background of a spinning extremal black hole are obtained by extremizing this entropy function that depends only on the parameters labelling the near horizon background and the electric and magnetic charges and the angular momentum carried by the black hole.

Interestingly, in all known cases, the appearance of flat directions in the entropy is associated with the presence of an ergo-sphere. Since not all moduli are fixed at the horizon the mass is not guaranteed to be fixed. The microscopic analysis of [3] confirms that in fact the mass is not fixed and so there is a nice microscopic interpretation for the ergo-branch. On the other hand, the slowly spinning extremal black holes in the ergo-free branch lack the rotational superradiance but can produce superradiant amplification of KK electric charged waves. However, this phenomenon can not be easily seen in the CFT since it is related to a modification of the central charge.

One important question is if there is a similar effective potential for stationary black holes and if one can use a similar analysis as in the static case to study the attractor mechanism. Unfortunately, at this point, we have just shown that the equations of motion at the horizon decouple from the bulk — we hope to report a detailed analysis elsewhere. Here, let us just indicate the main step in this analysis. We start by trying to solve the equations of motion and Bianchi identities for the gauge fields. However, unlike the static case, we can not obtain a general expression for the gauge fields, but rather their expressions in terms of two unknown functions (A and u):

$$F_{rt}^M = \partial_r A_t^M(r, \theta) \qquad F_{\theta t}^M = \partial_\theta A_t^M(r, \theta)$$

$$F^{M r \phi} = f^{MN}(\phi^i) \frac{\partial_\theta u_N(r, \theta)}{\sqrt{-g}} \quad F^{M \theta \phi} = -f^{MN}(\phi^i) \frac{\partial_r u_N(r, \theta)}{\sqrt{-g}} \quad (43)$$

One can write down the expressions of the gauge fields in some concrete examples. However, an interesting exercise is to work with this general form of the gauge fields and try to extract as much information as possible from the equations of motion. For the moduli the equations of motion become

$$\begin{aligned} \frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} \partial^\mu \phi^i) &= \frac{1}{2} \frac{\partial f_{MN}}{\partial \phi^i} \left(\partial_r A^M \partial_r A^N + r^2 \partial_\theta A^M \partial_\theta A^N \right) \\ &+ \frac{1}{2} \frac{\partial f^{MN}}{\partial \phi^i} \left(\partial_r u_M \partial_r u_N + r^2 \partial_\theta u_M \partial_\theta u_N \right) \end{aligned} \quad (44)$$

However, in this case, the moduli have also an angular dependence and the equations do not decouple. In principle one should be able to read off the effective potential from the right hand side of this equation, but that is not straightforward in this case — an effective potential for constant scalar fields was proposed in [9]. The best thing we can do is to check what is happening in the near horizon limit. After some tedious manipulations we found that in the near horizon limit the moduli equations are decoupled from the bulk. The scalar fields at the horizon have also an angular dependence and we obtain a system of distributions rather than functions. Thus, the boundary conditions, i.e. the values of the fields at the poles of the horizon, are important and the equations are difficult to be solved in a general case — concrete examples are presented in [9].

5 Discussion

Recently, after the proposal of Sen [10], there was a lot of work on attractor mechanism and entropy function (see, e.g., [19]). Motivated by the generalization of the attractor mechanism to non-supersymmetric extremal stationary black holes, we investigated the near horizon geometry of spinning extremal black holes in a theory of gravity with neutral scalar fields non-minimally coupled to abelian gauge fields. We found that the near horizon geometry of these black holes has the symmetry of $AdS_2 \times S^1$ — the AdS_2 part does not decouple from the angular part. Consequently, the horizons are attractors for the moduli and their geometry is independent of the boundary moduli data. One subtlety is that the extremal spinning black holes are further divided in two branches: ergo- and ergo-free branch, respectively. In both cases the $SO(2,1)$ isometry of AdS_2 is generated by the Killing vectors:

$$L_1 = \partial_t, \quad L_0 = t \partial_t - r \partial_r, \quad L_{-1} = (1/2)(1/r^2 + t^2) \partial_t - (tr) \partial_r + (\eta/r) \partial_\phi, \quad (45)$$

but they have distinct properties. The former is characterized by an entropy function with flat directions and for the latter there is no flat directions of the entropy function. If there are no flat directions then, clearly, the entropy is independent of the moduli. On the other hand,

if there are flat directions, then the extremization of the entropy function does not determine all the moduli values at the horizon. Location of these parameters along the flat directions may depend on the asymptotic values of the moduli. But since the entropy function does not depend on the flat directions, the entropy is still independent of the asymptotic values of the moduli, and so has an attractor behaviour.

Let us comment now on the physics of the two branches. In general, in the supergravity approximation, the entropy is a function of the duality invariant combinations $D(Q_A, P^B)$, $S = \sqrt{\pm(|D| - J^2)}$ and the mass saturates an extremality bound that is independent of the angular momentum parameter, J^2 — the plus sign corresponds to the ergo-free branch and the minus sign to the ergo-branch. When $D = J^2$ the extremal horizon disappears and becomes a naked singularity — this situation resembles the static case with one charge. Except this situation, the extremal limit has finite area and zero surface gravity. The fastly spinning extremal black holes have a non-zero horizon angular velocity and so their causal structure is similar with Kerr solution. Let us start with a non-extremal black hole that Hawking radiates. Clearly, Hawking radiation carries away the angular momentum and so the black hole is slowing down. If the black hole is radiating away all the angular momentum before reaching the extremal limit, then the corresponding solution will be in the ergo-free branch. On the other hand, if the black hole reaches the extremal limit and the angular velocity is non-zero, then there is radiation due to the ergo-region. If the evaporating process is fine tuned such that the extremal limit is reached when $J^2 = |D|$, then the black hole behaves more as an elementary particle [20] — there are potential barriers outside the horizon which increase without bound.

In this paper we also tried to extend the analysis of the effective potential to extremal spinning black holes. We have not been able to conclusively construct an *explicit* effective potential, mainly because of technical obstacles. In the static case one can explicitly check that the moduli are fixed at the attractor horizon that is a critical point for the effective potential. A similar analysis is difficult for the stationary case. However, by studying the equations of motion for the moduli, we concluded that they decouple from the bulk at the horizon.⁴ A complete determination of the scalar fields at the horizon needs also imposing the boundary conditions which are the values of the fields at the poles of the horizon.

The near horizon geometry of a stationary extremal black hole is universal and so the entropy does not depend of couplings. The extremality condition is very powerful to force an attractor behaviour of the horizon — it is independent of the supersymmetry of the theory/solution. This does not come as a surprise, though, since the near horizon geometry has an enhanced symmetry and the long throat of AdS_2 is the main ingredient for the existence of the attractor mechanism.

⁴This is not a sufficient condition for the attractor mechanism to exist. However, a rigorous proof was given in [9] by using the entropy function formalism.

Acknowledgements

We thank Kevin Goldstein for collaboration in the initial stages of this work and for further discussions. It is also a pleasure to thank Soo-Jong Rey, Ashoke Sen, and Sandip Trivedi for useful conversations. DA would like to thank KIAS, Seoul for hospitality during part of this work. DA has presented this work at ISM06 Puri (December 2006), KIAS, Seoul (February 2007), YITP, Kyoto (February, 2007), TITECH, Tokyo (May 2007) and he likes to thank the audience at all these places for their positive feedback. The work of DA has been done with support from MEXT's program "Promotion of Environmental Improvement for Independence of Young Researchers" under the Special Coordination Funds for Promoting Science and Technology, Japan. DA also acknowledges support from NSERC of Canada. HY would like to thank the Korea Research Foundation Leading Scientist Grant (R02-2004-000-10150-0) and Star Faculty Grant (KRF-2005-084-C00003). While this paper was being completed, ref. [21] appeared which overlaps with the material presented in section 3.

A The Maxwell equations in the near horizon limit

In this appendix we explicitly obtain the equations of motion for the gauge fields in the near horizon limit. These expressions are useful in subsection (3.2) — for simplicity, we specialize again to a configuration with one scalar and one gauge field, but the generalization is straightforward.

The non-zero components of the Maxwell tensor are given by

$$\left. \begin{aligned} F_{ti} &= (a_t(\theta), r a'_t(\theta), 0) \\ F_{i\theta} &= (a'_r(\theta)/r, 0, a'_\phi(\theta)) \end{aligned} \right\} \quad \text{where } i \in \{r, \theta, \phi\} \quad (46)$$

Raising the indices we obtain

$$F^{ti} = -\frac{C^2}{\mu^2(\theta)} \left(a_t(\theta), \frac{\zeta(\theta)}{r}, 0 \right) \quad (47)$$

$$F^{i\theta} = \frac{C^2}{\mu^2(\theta)} \left(C^2 r a'_r(\theta), 0, \frac{\mu^2(\theta)}{\sin^2 \theta} a'_\phi(\theta) + \eta(\theta) \zeta(\theta) \right) \quad (48)$$

$$F^{\phi r} = \frac{C^2}{\mu^2(\theta)} \eta(\theta) r a_t(\theta) \quad (49)$$

where $\zeta(\theta) = a'_t(\theta) - \eta(\theta) a'_\phi(\theta)$.

Maxwell's equations are

$$\Phi_\mu(\sqrt{-G} f F^{\mu\nu}) = 0 \quad (50)$$

From the r -component of Maxwell's equation we obtain

$$\Phi_\theta(\mu^{-1}(\theta) \sin(\theta) f(\theta) a'_r(\theta)) = 0 \quad (51)$$

which can be integrated to give

$$a'_r(\theta) = \kappa_1 \frac{\mu(\theta)}{f \sin \theta} \sim \frac{\kappa_1}{\theta} \quad (52)$$

where we have assumed that the effective gauge coupling at the north pole, $f(\theta = 0)$, is well behaved. Now, for F^2 to be finite at $\theta = 0$, we require $\kappa_1 = 0$, i.e. $a'_r = 0$. This in turn means that $a_r(\theta)$ does not contribute to the Maxwell tensor and can be gauged away.

Similarly from the t -component of Maxwell's equation we obtain

$$\Phi_\theta(\mu^{-1} \sin(\theta) f(\theta) \zeta(\theta)) = 0 \quad (53)$$

which, by an argument similar to the one for a'_r above, implies ζ is zero. Some important relations used in this derivation are

$$\sqrt{-G} = C^{-2} \mu(\theta) \sin \theta \quad (54)$$

and

$$\begin{aligned} (\Phi_s)^2 &= g^{\mu\nu} \Phi_\mu \Phi_\nu \\ &= \frac{1}{\mu(\theta)} \left(-[r^{-1} \Phi_t - \eta \Phi_\phi]^2 + C^2 r^2 \Phi_r^2 + C^2 \Phi_\theta^2 \right) + \frac{\mu(\theta)}{\sin^2 \theta} \Phi_\phi^2 \end{aligned} \quad (55)$$

$$F^2 = \frac{2C^2}{\mu^2} \left(-a_t^2 - \zeta^2 + C^2 a_r^2 \right) + \frac{2C^2}{\sin^2 \theta} (a'_\phi)^2 \quad (56)$$

References

- [1] A. Dabholkar, A. Sen and S. Trivedi, “Black hole microstates and attractor without supersymmetry,” arXiv:hep-th/0611143.
- [2] D. Astefanesei, K. Goldstein and S. Mahapatra, “Moduli and (un)attractor black hole thermodynamics,” arXiv:hep-th/0611140.
- [3] R. Emparan and A. Maccarone “Statistical description of rotating Kaluza-Klein black hole” Phys. Rev. D **75**, 084006 (2007) arXiv:hep-th/0701150
- [4] R. Emparan and G. T. Horowitz, “Microstates of a neutral black hole in M theory,” Phys. Rev. Lett. **97** (2006) 141601 [arXiv:hep-th/0607023].
- [5] G. T. Horowitz, D. A. Lowe and J. M. Maldacena, “Statistical Entropy of Nonextremal Four-Dimensional Black Holes and U-Duality,” Phys. Rev. Lett. **77**, 430 (1996) [arXiv:hep-th/9603195].
- [6] J. M. Maldacena and A. Strominger, “Statistical Entropy of Four-Dimensional Extremal Black Holes,” Phys. Rev. Lett. **77**, 428 (1996) [arXiv:hep-th/9603060];

- [7] A. Dabholkar, “Microstates of non-supersymmetric black holes,” *Phys. Lett. B* **402**, 53 (1997) [arXiv:hep-th/9702050].
- [8] J. M. Bardeen and G. T. Horowitz, *Phys. Rev. D* **60**, 104030 (1999) [arXiv:hep-th/9905099].
- [9] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, “Rotating attractors,” *JHEP* **0610** (2006) 058 [arXiv:hep-th/0606244].
- [10] A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” *JHEP* **0509**, 038 (2005) [arXiv:hep-th/0506177].
- [11] D. Rasheed, “The Rotating dyonic black holes of Kaluza-Klein theory,” *Nucl. Phys. B* **454**, 379 (1995) [arXiv:hep-th/9505038].
- [12] S. Hollands, A. Ishibashi, and R. M. Wald, “A higher dimensional stationary rotating black hole must be axisymmetric,” *Commun. Math. Phys.* **271**, 699 (2007) [arxiv:gr-qc/0605106].
- [13] D. Astefanesei and E. Radu, “Quasilocal formalism and black ring thermodynamics,” *Phys. Rev. D* **73**, 044014 (2006) [arXiv:hep-th/0509144].
- [14] R. B. Mann and D. Marolf, “Holographic renormalization of asymptotically flat spacetimes,” *Class. Quant. Grav.* **23**, 2927 (2006) [arXiv:hep-th/0511096].
- [15] D. Astefanesei, R. B. Mann and C. Stelea, “Note on counterterms in asymptotically flat spacetimes,” arXiv:hep-th/0608037.
- [16] D. Astefanesei, R. B. Mann, and C. Stelea, to appear.
- [17] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, “Non-supersymmetric attractors,” *Phys. Rev. D* **72**, 124021 (2005) [arXiv:hep-th/0507096].
- [18] S. Ferrara, G. W. Gibbons, and R. Kallosh, “Black holes and critical points in moduli space,” *Nucl. Phys. B* **500**, 75 (1997) [arxiv:hep-th/9702103].
- [19] P. K. Tripathy and S. P. Trivedi, “Non-supersymmetric attractors in string theory,” *JHEP* **0603**, 022 (2006) [arXiv:hep-th/0511117]. M. Alishahiha and H. Ebrahim, “Non-supersymmetric attractors and entropy function,” *JHEP* **0603**, 003 (2006) [arXiv:hep-th/0601016]. R. Kallosh, N. Sivanandam and M. Soroush, “The non-BPS black hole attractor equation,” *JHEP* **0603**, 060 (2006) [arXiv:hep-th/0602005]. B. Chandrasekhar, S. Parvizi, A. Tavanfar and H. Yavartanoo, “Non-supersymmetric attractors in R^{*2} gravities,” *JHEP* **0608**, 004 (2006) [arXiv:hep-th/0602022]. S. Bellucci, S. Ferrara and A. Marrani, “On some properties of the attractor equations,” *Phys. Lett. B* **635**, 172 (2006) [arXiv:hep-th/0602161]. S. Ferrara and R. Kallosh, “On $N = 8$ attractors,” *Phys. Rev. D* **73**, 125005 (2006) [arXiv:hep-th/0603247]. M. Alishahiha and H. Ebrahim, “New attractor, entropy function and black hole partition function,” *JHEP*

0611, 017 (2006) [arXiv:hep-th/0605279]. S. Kar and S. Majumdar, “Noncommutative D(3)-brane, black holes and attractor mechanism,” Phys. Rev. D **74**, 066003 (2006) [arXiv:hep-th/0606026]. S. Ferrara and M. Gunaydin, “Orbits and attractors for $N = 2$ Maxwell-Einstein supergravity theories in five dimensions,” Nucl. Phys. B **759**, 1 (2006) [arXiv:hep-th/0606108]. P. Kaura and A. Misra, “On the existence of non-supersymmetric black hole attractors for two-parameter Calabi-Yau’s and attractor equations,” Fortsch. Phys. **54**, 1109 (2006) [arXiv:hep-th/0607132]. G. L. Cardoso, V. Grass, D. Lust and J. Perz, “Extremal non-BPS black holes and entropy extremization,” JHEP **0609**, 078 (2006) [arXiv:hep-th/0607202]. H. Arfaei and R. Fareghbal, “Double-horizon limit and decoupling of the dynamics at the horizon,” JHEP **0701**, 060 (2007) [arXiv:hep-th/0608222]. B. Chandrasekhar, H. Yavartanoo and S. Yun, “Non-Supersymmetric attractors in BI black holes,” arXiv:hep-th/0611240. G. L. Cardoso, B. de Wit and S. Mahapatra, “Black hole entropy functions and attractor equations,” arXiv:hep-th/0612225. R. D’Auria, S. Ferrara and M. Trigiante, “Critical points of the black-hole potential for homogeneous special geometries,” arXiv:hep-th/0701090. S. Bellucci, S. Ferrara and A. Marrani, “Attractor horizon geometries of extremal black holes,” arXiv:hep-th/0702019. A. Sen, “Entropy function for heterotic black holes,” JHEP **0603**, 008 (2006) [arXiv:hep-th/0508042]. A. Ghodsi, “ R^4 corrections to D1D5p black hole entropy from entropy function formalism,” Phys. Rev. D **74**, 124026 (2006) [arXiv:hep-th/0604106]. P. Prester, “Love-lock type gravity and small black holes in heterotic string theory,” JHEP **0602**, 039 (2006) [arXiv:hep-th/0511306]. A. Sinha and N. V. Suryanarayana, “Extremal single-charge small black holes: Entropy function analysis,” Class. Quant. Grav. **23**, 3305 (2006) [arXiv:hep-th/0601183]. B. Sahoo and A. Sen, “Higher derivative corrections to non-supersymmetric extremal black holes in JHEP **0609**, 029 (2006) [arXiv:hep-th/0603149]. B. Chandrasekhar, “Born-Infeld corrections to the entropy function of heterotic black holes,” arXiv:hep-th/0604028. R. G. Cai and D. W. Pang, “Entropy function for 4-charge extremal black holes in type IIA superstring theory,” Phys. Rev. D **74**, 064031 (2006) [arXiv:hep-th/0606098]. G. L. Cardoso, J. M. Oberreuter and J. Perz, “Entropy function for rotating extremal black holes in very special geometry,” arXiv:hep-th/0701176. S. Nam-puri, P. K. Tripathy, and S. Trivedi “On the stability of non-supersymmetric attractors in string theory,” arXiv: 0705.4554 [hep-th]. A. Ceresole and G. Dall’Agata, “Flow equations for non-BPS extremal black holes,” JHEP **0703**, 110 (2007) [arXiv:hep-th/0702088]. L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, “Black-hole attractors in $N = 1$ supergravity,” arXiv:hep-th/0703178. K. Saraikin and C. Vafa, “Non-supersymmetric black holes and topological strings,” arXiv:hep-th/0703214. M. R. Garousi and A. Ghodsi, “On Attractor Mechanism and Entropy Function for Non-extremal Black Holes/Branes,” arXiv:hep-th/0703260. J. H. Cho and S. Nam, “Non-supersymmetric Attractor with the Cosmological Constant,” arXiv:0705.2892 [hep-th]. M. Cvitan, P. D. Prester, S. Pal-lua, I. Smolic “Extremal black holes in $D=5$: SUSY vs Gauss-Bonnet corrections” arXiv:0706.1167 [hep-th]. R. G. Cai and L. M. Cao, “On the entropy function and the attractor mechanism for spherically symmetric extremal black holes,” arXiv:0704.1239 [hep-th]. O. Dias and P. J. Silva, “Attractors and the quantum statistical relation for extreme (BPS or not) black holes,” arXiv:0704.1405 [hep-th]. N. V. Suryanarayana and

- M. Wapler, “Charges from attractors,” arXiv:0704.0955 [hep-th] Y. S. Myung, Y. W. Kim, and Y. J. Park, “New attractor mechanism for spherically symmetric extremal black holes,” [arXiv:hep-th/0707.1933]. Y. S. Myung, Y.-W. Kim, and Y.-J. Park, “Entropy of an extremal regular black hole”, [arXiv:gr-qc/0705.2478]. S. Hyun, W. Kim, J. J. Oh, J. Son “Entropy function and universal entropy of two-dimensional extremal black holes” JHEP **0704**, 057 (2007) [arXiv:hep-th/0702170].
- [20] C. Holzhey and F. Wilczek “Black holes as elementary particles,” Nucl. Phys. B **380** 447 (1992) [aXiv: hep-th/9202014]
- [21] H. K. Kunduri, J. Lucietti, and H. Reall “Near-horizon symmetries of extremal black holes” [arXiv:0705.4214]